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R. Loudon

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Propagation of non-classical light

By R. Loudon
Department of Physics, University of Essex, Colchester CO4 3SQ, UK

The theoretical descriptions of the main varieties of non-classical light and the principal measures of their specifically quantum properties are outlined. The theories of quantum-optical propagation of the light through slabs of amplifying or attenuating materials are presented and the main features of the quantum noise effects produced by transmission are summarized. It is found that, while quantum light may have lower noise than classical coherent light, transmission through the slab generally increases the noise. Thus initial enthusiasm for the practical low-noise advantages of non-classical light has been tempered by the realization that most kinds of optical propagation or processing enhance the noise to near-classical levels.

1. Introduction

Formulations of the theoretical foundations of quantum optics have mainly assumed a radiation field in free space excited by, or interacting with, isolated atoms or optical components whose material properties are not important for the observed effects. Such theoretical models are in close correspondence with some of the key experiments on the generation and properties of non-classical light; for example, cascade emission of photon pairs (Clauser 1974), resonance fluorescence by laser excitation of an atomic beam (Kimble *et al.* 1978), the generation of squeezed light in an atomic vapour (Slusher *et al.* 1985), single-photon interference (Grangier *et al.* 1986) and two-photon interference (Hong *et al.* 1987). Nevertheless, there are other quantum-optical effects that involve the propagation of non-classical light beams on to the surfaces, or through the bulk, of dielectric media or the interactions of light with atoms embedded in, or adjacent to, materials whose optical properties are important for the outcomes of appropriate experiments. Such processes include spontaneous emission by atoms close to or inside media, the propagation of non-classical light through absorbing or amplifying media, and the radiation pressure exerted on media by incident beams of non-classical light.

A recent focus of interest in quantum optics theory has been the development of quantization schemes for optical systems that include regions of dielectric material. Such systems are employed in some of the proposed applications that could take advantage of the reduced noise available in non-classical light. Ideally, one needs a quantum-mechanical formalism for the propagation of electromagnetic fields in three dimensions in environments that include optical components of finite extent made from lossy or amplifying media. Theories of such generality are currently emergent for some simple environments. However, the present paper is restricted to problems of propagation in which a single dimension is selected by a plane parallel light beam in normal incidence from free space on the surfaces of material components. Intrinsically

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three-dimensional phenomena, such as transverse effects or spontaneous emission, are thus excluded from consideration. The main varieties of non-classical light and the main indicators of its quantum properties are summarized in § 2. The effects of propagation on the non-classical properties are discussed in § 3.

2. Photon states

(a) Photon number states

Practical photon number states have the form of a wavepacket, whose amplitude spectrum is represented by a function $\alpha(\omega)$ determined by the nature of the light source and any subsequent filtering. The theory is conveniently illustrated by a Gaussian spectrum, with

$$\alpha(\omega) = \left(\frac{L^2}{2\pi c^2}\right)^{1/4} \exp\left\{ \mathrm{i}(\omega - \omega_\mathrm{c})t_0 - \frac{L^2(\omega - \omega_\mathrm{c})^2}{4c^2} \right\}, \tag{2.1}$$

where ω_c is the carrier frequency, L is the pulse length and c/L is the root-mean-square spread of the intensity spectrum; t_0 is a reference time at which the peak of the wavepacket passes the coordinate origin.

The creation operator for the corresponding photon-number states is (Collett 1984, unpublished; Blow *et al.* 1990)

$$\hat{A}^\dagger(\alpha) = \int \mathrm{d}\omega\, \alpha(\omega)\hat{a}^\dagger(\omega), \tag{2.2}$$

where $\hat{a}^\dagger(\omega)$ is the usual continuous-mode creation operator with commutator

$$[\hat{a}(\omega), \hat{a}^\dagger(\omega')] = \delta(\omega - \omega'). \tag{2.3}$$

The n-photon state is

$$|n(\alpha)\rangle = \frac{1}{\sqrt{n!}}[\hat{A}^\dagger(\alpha)]^n|0\rangle, \tag{2.4}$$

where $|0\rangle$ is the universal vacuum state. The number states are normalized for wavepacket spectra that satisfy

$$\int \mathrm{d}\omega\, |\alpha(\omega)|^2 = 1. \tag{2.5}$$

It is often useful to work in the time domain and, although the frequency is strictly a positive quantity, it is permissible to define Fourier transformed operators

$$\hat{a}(t) = (2\pi)^{-1/2} \int \mathrm{d}\omega\, \hat{a}(\omega) \exp(-\mathrm{i}\omega t), \tag{2.6}$$

whose application is restricted to states with narrow bandwidths, that is, $c/L \ll \omega_\mathrm{c}$ for the spectrum $\alpha(\omega)$ in (2.1). The transform of this spectrum is the wavepacket amplitude

$$\tilde{\alpha}(t) = \left(\frac{2c^2}{\pi L^2}\right)^{1/4} \exp\left\{ -\mathrm{i}\omega_\mathrm{c} t - \frac{c^2(t - t_0)^2}{L^2} \right\}, \tag{2.7}$$

and (2.2)–(2.5) can all be written in equivalent time-domain forms. Free-space propagation of the wavepacket along the x-axis is simply inserted by the replacement $t \to t - (x/c)$.

The optical flux in units of photons per unit time is represented by the operator

$$\hat{f}(t) = \hat{a}^\dagger(t)\hat{a}(t), \tag{2.8}$$

whose mean value for the number state (2.4) is

$$f(t) = \langle \hat{f}(t)\rangle = n|\tilde{\alpha}(t)|^2. \tag{2.9}$$

The total number of photons in the system is represented by the dimensionless operator

$$\hat{N} = \int \mathrm{d}t\, \hat{f}(t) = \int \mathrm{d}t\, \hat{a}^\dagger(t)\hat{a}(t) = \int \mathrm{d}\omega\, \hat{a}^\dagger(\omega)\hat{a}(\omega), \tag{2.10}$$

with the eigenvalue property

$$\hat{N}|n(\alpha)\rangle = n|n(\alpha)\rangle. \tag{2.11}$$

A stationary light beam has a mean flux $f(t)$ that is independent of the time. The photon number state cannot be stationary as this would imply functions $\alpha(\omega)$ or $\alpha(t)$ that could not be normalized in accordance with (2.5).

The degree of second-order coherence is defined as

$$g^{(2)}(t,\tau) = \frac{\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle\langle \hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\rangle} \tag{2.12}$$

and its value for the number state (2.4) is

$$g^{(2)}(t,\tau) = 1 - 1/n, \tag{2.13}$$

which is identical to the standard result for the single-mode number state and is independent of t and τ, despite the time dependence of the wavepacket amplitude (2.7). Classical light beams have degrees of second-order coherence that satisfy

$$g^{(2)}(t,0) \geqslant 1, \tag{2.14}$$

and the photon number state clearly violates this inequality. The violation of (2.14) represents a form of the so-called photon antibunching.

(*b*) *Coherent states*

The coherent states are defined in terms of the operator (2.2) as

$$|\{\alpha\}\rangle = \exp\{\hat{A}^\dagger(\alpha) - \hat{A}(\alpha)\}|0\rangle. \tag{2.15}$$

The coherent states are automatically normalized for any spectrum $\alpha(\omega)$, and the normalization condition (2.5) no longer applies. They have the eigenvalue properties

$$\hat{a}(\omega)|\{\alpha\}\rangle = \alpha(\omega)|\{\alpha\}\rangle \quad \text{and} \quad \hat{a}(t)|\{\alpha\}\rangle = \alpha(t)|\{\alpha\}\rangle. \tag{2.16}$$

The mean photon flux and number are

$$f(t) = |\tilde{\alpha}(t)|^2 \quad \text{and} \quad N = \int \mathrm{d}t\, |\tilde{\alpha}(t)|^2, \tag{2.17}$$

where the wavepacket amplitude may, for example, have a Gaussian form similar to (2.7). Another simple example is the 'single-mode' coherent state represented by

$$\tilde{\alpha}(t) = \sqrt{F}\exp(-\mathrm{i}\omega_\mathrm{c}t + \mathrm{i}\vartheta) \quad \text{and} \quad \alpha(\omega) = \sqrt{2\pi F}\exp(\mathrm{i}\vartheta)\delta(\omega - \omega_\mathrm{c}), \tag{2.18}$$

where ϑ is the phase and F is the time-independent mean flux. This represents a stationary light beam with an infinite mean photon number but, in contrast to the photon number state, no violation of normalizability occurs in this case.

The coherent states are linear superpositions of photon-number states, obtained by expansion of the exponent in (2.15). The superposition removes any non-classical features from the coherent states, whose degree of second-order coherence has the usual value of unity.

(c) Photon pair states

Consider now two independent sets of continuous field modes. Various continua are of practical interest, but we take the example of light beams propagating along the same x-axis but with orthogonal polarizations, denoted $\parallel$ and $\perp$. The creation operator for a pair state in which photons with correlated frequencies are excited, one of each polarization, is

$$\hat{P}^\dagger(\beta_{\parallel,\perp}) = \int \mathrm{d}\omega \int \mathrm{d}\omega'\, \beta_{\parallel,\perp}(\omega,\omega')\hat{a}^\dagger_\parallel(\omega)\hat{a}^\dagger_\perp(\omega'), \tag{2.19}$$

and the photon pair state is defined by

$$|1(\beta_{\parallel,\perp})\rangle = \hat{P}^\dagger(\beta_{\parallel,\perp})|0\rangle. \tag{2.20}$$

The joint two-photon wavepacket amplitude $\beta_{\parallel,\perp}(\omega,\omega')$ has the normalization

$$\int \mathrm{d}\omega \int \mathrm{d}\omega'\, |\beta_{\parallel,\perp}(\omega,\omega')|^2 = 1. \tag{2.21}$$

The expressions (2.19) and (2.21) have the same forms in the time domain, with ω merely replaced by t. The pair state has the eigenvalue properties

$$\hat{N}_\parallel|1(\beta_{\parallel,\perp})\rangle = \hat{N}_\perp|1(\beta_{\parallel,\perp})\rangle = |1(\beta_{\parallel,\perp})\rangle. \tag{2.22}$$

The mean photon fluxes for the two polarizations are

$$f_\parallel(t) = \int \mathrm{d}t'\, |\beta_{\parallel,\perp}(t,t')|^2 \quad \text{and} \quad f_\perp(t) = \int \mathrm{d}t'\, |\beta_{\parallel,\perp}(t',t)|^2. \tag{2.23}$$

No stationary pair state is possible, as the corresponding wavepacket amplitude would not be normalizable.

The photon pair state has several non-classical features. Thus the degrees of second-order coherence defined as in (2.12) vanish when all the operators refer to a single polarization

$$g^{(2)}_{\parallel,\parallel}(t,\tau) = g^{(2)}_{\perp,\perp}(t,\tau) = 0, \tag{2.24}$$

in agreement with (2.13) as the photon number equals unity for each polarization. However, the cross-polarization degree of second-order coherence does not vanish,

$$g^{(2)}_{\parallel,\perp}(t,\tau) = \frac{|\beta_{\parallel,\perp}(t,t+\tau)|^2}{f_\parallel(t)f_\perp(t+\tau)}, \tag{2.25}$$

in accordance with the presence of one photon of each polarization. The vanishing of (2.24) but not of (2.25) violates the inequality

$$[g^{(2)}_{\parallel,\perp}(t,\tau)]^2 \leqslant g^{(2)}_{\parallel,\parallel}(t,0)g^{(2)}_{\perp,\perp}(t,0) \tag{2.26}$$

that must be satisfied by the degrees of second-order coherence for intensity measurements on classical light beams.

(*d*) *Two-photon coherent states*

The two-photon coherent states are defined in terms of the pair operator (2.19) as

$$|\{\{\beta_{\|,\perp}\}\}\rangle = \exp\{\hat{P}(\beta_{\|,\perp}) - \hat{P}^\dagger(\beta_{\|,\perp})\}|0\rangle, \tag{2.27}$$

analogous to the single-photon coherent states (2.15) but with a conventional change in the sign of the exponent. The states are automatically normalized and there is no need for $\beta_{\|,\perp}(\omega,\omega')$ to satisfy (2.21). It is possible to define a stationary light beam, such as that generated by parametric down conversion with a 'single-mode' pump of frequency ω_p, where

$$\beta_{\|,\perp}(\omega,\omega') = \beta(\omega)\delta(\omega+\omega'-\omega_\mathrm{p}), \quad 0 \leqslant \omega,\ \omega' \leqslant \omega_\mathrm{p} \tag{2.28}$$

and

$$\beta_{\|,\perp}(t,t') = (2\pi)^{-1/2}\tilde{\beta}(t-t')\exp(-\mathrm{i}\omega_\mathrm{p}t'). \tag{2.29}$$

The mean fluxes in the two beams are (Blow *et al.* 1990)

$$f_\|(t) = f_\perp(t) = (2\pi)^{-1}\int_0^{\omega_\mathrm{p}} \mathrm{d}\omega\ \sinh^2|\beta(\omega)|. \tag{2.30}$$

The non-classical properties of the single pair state are lost to some extent by the formation of the superposition of multiple pair states given by (2.27). Thus the degrees of second-order coherence for the individual polarizations have the forms

$$g^{(2)}_{\|,\|}(\tau) = g^{(2)}_{\perp,\perp}(\tau) = 1 + \frac{\left|\int \mathrm{d}\omega\ \exp(-\mathrm{i}\omega\tau)\sinh^2|\beta(\omega)|\right|^2}{\left[\int \mathrm{d}\omega\ \sinh^2|\beta(\omega)|\right]^2} \tag{2.31}$$

characteristic of chaotic light (Barnett & Knight 1985). However, the cross-polarization degree of second-order coherence

$$g^{(2)}_{\|,\perp}(\tau) = 1 + \frac{\left|\int \mathrm{d}\omega\ \exp[-\mathrm{i}\omega\tau - \mathrm{i}\arg\beta(\omega)]\sinh|\beta(\omega)|\cosh|\beta(\omega)|\right|^2}{\left[\int \mathrm{d}\omega\ \sinh^2|\beta(\omega)|\right]^2}, \tag{2.32}$$

continues to violate the classical inequality (2.26) as $\cosh|\beta(\omega)|$ is larger than $\sinh|\beta(\omega)|$.

(*e*) *Squeezed states*

The operation on the right of (2.27) generates squeezed vacuum states, denoted $|\{\{\beta\}\}\rangle$, when the two operators in the pair creation integrand of (2.19) refer to a single set of continuous modes. The substitution (2.28) continues to define a stationary light beam, such as that produced by degenerate parametric down conversion, and this special case is assumed here. Some of the properties of the squeezed states are obtained by simple removal of the polarization subscripts from results given in § 2 *d*, as in the expression (2.30) for the flux in the single light beam. However, the expression (2.31) for the intrabeam degree of second-order coherence is no longer correct, and of course the interbeam result (2.32) becomes meaningless. The second-order coherence of squeezed light is not needed for our subsequent calculations.

The expectation values of products of pairs of field operators are particularly important for applications of squeezed light, and they are

$$\langle \hat{a}^\dagger(\omega)\hat{a}(\omega')\rangle = \sinh^2|\beta(\omega)|\delta(\omega-\omega') \tag{2.33}$$

and

$$\langle \hat{a}^\dagger(\omega)\hat{a}^\dagger(\omega')\rangle = \tfrac{1}{2}\exp[-\mathrm{i}\arg\beta(\omega)]\sinh[2|\beta(\omega)|]\delta(\omega+\omega'-\omega_\mathrm{p}). \tag{2.34}$$

A homodyne measurement of the electric field of the light is represented by the operator

$$\hat{E}_\mathrm{LO} = \frac{\exp(\mathrm{i}\phi_\mathrm{LO})}{\sqrt{2\pi T_0}}\int \mathrm{d}\omega\left\{\hat{a}^\dagger(\omega)\frac{\exp[\mathrm{i}(\omega-\omega_\mathrm{LO})T_0]-1}{\omega-\omega_\mathrm{LO}}+\mathrm{H.c.}\right\}, \tag{2.35}$$

where ω_LO is the local oscillator frequency, ϕ_LO is its phase and T_0 is the duration of the measurement. The normalization of the homodyne field operator is such that its variance equals unity for the vacuum state or for coherent light, and another condition satisfied by classical light is the inequality

$$\langle(\Delta E[\phi_\mathrm{LO}])^2\rangle \geqslant 1. \tag{2.36}$$

With a local oscillator frequency equal to the central frequency $\frac{1}{2}\omega_\mathrm{p}$ of the squeezing and an integration time T_0 sufficiently long that $\beta(\omega)$ varies little over the frequency range $1/T_0$, the variance of the homodyne field for the squeezed vacuum state is given approximately by

$$\begin{aligned}\langle(\Delta E[\phi_\mathrm{LO}])^2\rangle = \exp[2|\beta(\omega_\mathrm{LO})|]\sin^2[\phi_\mathrm{LO}-\tfrac{1}{2}\arg\beta(\omega_\mathrm{LO})]\\ +\exp[-2|\beta(\omega_\mathrm{LO})|]\cos^2[\phi_\mathrm{LO}-\tfrac{1}{2}\arg\beta(\omega_\mathrm{LO})].\end{aligned} \tag{2.37}$$

This reduces to the vacuum value for $\beta(\omega_\mathrm{LO}) = 0$ but it takes non-classical values smaller than unity for $|\beta(\omega_\mathrm{LO})| > 0$ and for appropriate values of the local oscillator phase. The amount of squeezing is conveniently characterized by the reduction of the variance below unity.

3. Propagation through a material slab

Non-classical light that violates the inequalities (2.14) or (2.36) corresponds to fluctuation spectra or noise with values smaller than those of classical light beams, and various applications could, in principle, take advantage of the non-classical properties. We derive the extents to which the reduced noise survives transmission through a slab of amplifying or attenuating material.

(*a*) *Field quantization*

We treat the propagation of non-classical light through a dielectric slab defined by the dielectric function

$$\varepsilon(x,\omega) = \begin{cases} 1, & \text{for } |x| > l,\\ \varepsilon(\omega), & \text{for } |x| < l.\end{cases} \tag{3.1}$$

The material dielectric function $\varepsilon(\omega)$ is related to the complex refractive index $n(\omega)$ by

$$\varepsilon(\omega) = [n(\omega)]^2, \quad \text{where } n(\omega) = \eta(\omega)+\mathrm{i}\kappa(\omega). \tag{3.2}$$

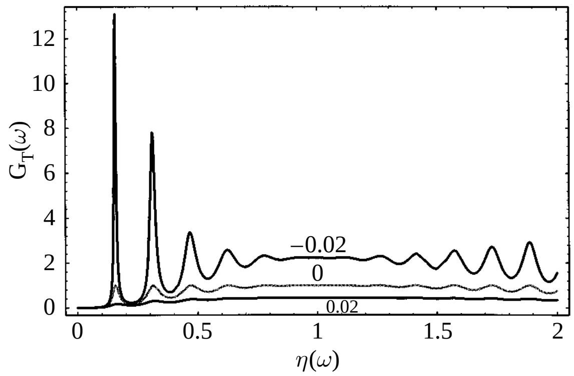

Figure 1. Transmission gain $G_T(\omega)$ of a dielectric slab of thickness $2l$ for a frequency $\omega = 10c/l$ as a function of the refractive index $\eta(\omega)$ for the values of extinction coefficient $\kappa(\omega)$ shown.

The reflection and transmission coefficients of the slab are

$$R(\omega) = -\frac{n(\omega)^2 - 1}{D(\omega)} \exp\left(-\frac{2\mathrm{i}\omega l}{c}\right)\left\{1 - \exp\left(\frac{4\mathrm{i}\omega n(\omega) l}{c}\right)\right\} \tag{3.3}$$

and

$$T(\omega) = \frac{4n(\omega)}{D(\omega)} \exp\left(\frac{2\mathrm{i}\omega(n(\omega) - 1)l}{c}\right), \tag{3.4}$$

where

$$D(\omega) = (n(\omega) + 1)^2 - (n(\omega) - 1)^2 \exp(4\mathrm{i}\omega n(\omega) l/c). \tag{3.5}$$

The extinction coefficient $\kappa(\omega)$ is positive for most frequencies ω, but it may be negative over limited ranges of frequency; the material attenuates or amplifies light of these respective frequencies. For the latter frequencies, reflection or transmission at the slab may produce amplification of an incident light beam with intensity gain coefficients defined by

$$G_R(\omega) = |R(\omega)|^2 \quad \text{and} \quad G_T(\omega) = |T(\omega)|^2. \tag{3.6}$$

For convenience, we use the term 'gain' even for positive $\kappa(\omega)$, where the light suffers a loss. These gains have the properties

$$G_R(\omega) + G_T(\omega) \begin{cases} > 1, & \text{for } \kappa(\omega) < 0 \text{ (gain)}, \\ = 1, & \text{for } \kappa(\omega) = 0, \\ < 1, & \text{for } \kappa(\omega) > 0 \text{ (loss)}, \end{cases} \tag{3.7}$$

where frequencies that satisfy the first and third conditions are termed 'amplified' and 'attenuated' frequencies, respectively. Figure 1 shows the form of the transmission gain as a function of the refractive index $\eta(\omega)$ for a fixed frequency and three values of $\kappa(\omega)$. The gain for $\kappa(\omega) = -0.02$ shows a strong resonance at $\eta(\omega) = 0.16$ associated with a pole at a nearby frequency where the amplifying slab has a lasing threshold. The theory presented here is valid for material parameters such that the conditions for laser oscillation are not satisfied for any frequency.

The notation for the destruction operators used in the electromagnetic field quantization for propagation parallel to the x-axis is shown in figure 2, where the subscripts on the operators indicate directions of propagation, *not* the two sides of the slab. The operators $\hat{a}_R(\omega)$, $\hat{b}_L(\omega)$ and $\hat{a}_L(\omega)$, $\hat{b}_R(\omega)$ represent the pairs of input and output

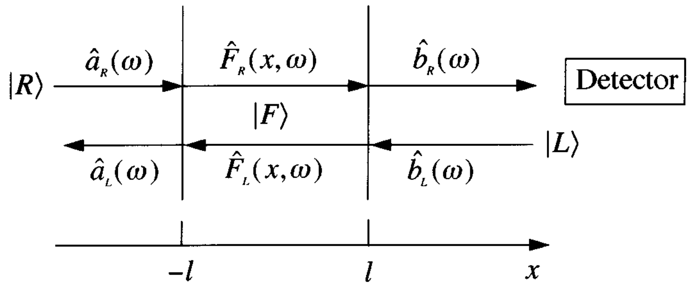

Figure 2. Spatial configuration of the dielectric slab with the notation for field quantization.

fields, respectively, and the states of the two input fields are denoted $|R\rangle$ and $|L\rangle$. The operators $\hat{F}_R(x,\omega)$ and $\hat{F}_L(x,\omega)$ represent the rightwards and leftwards propagating noise fields in the interior of the slab, and the state of the noise field is denoted by $|F\rangle$, although this strictly has the nature of a statistical mixture; the noise state can be described by an effective temperature, which takes negative values for amplified frequencies and positive values for attenuated frequencies. The relations between the output fields and the input and noise fields are

$$\hat{a}_L(\omega) = R(\omega)\hat{a}_R(\omega) + T(\omega)\hat{b}_L(\omega) + \hat{F}_L(\omega) \tag{3.8}$$

and

$$\hat{b}_R(\omega) = T(\omega)\hat{a}_R(\omega) + R(\omega)\hat{b}_L(\omega) + \hat{F}_R(\omega). \tag{3.9}$$

The coefficients of the first two terms on the right are consistent with the definitions of the intensity gains in (3.6). The final terms are noise operators (Matloob *et al.* 1995, 1997), proportional to $\sqrt{|\kappa(\omega)|}$, associated with the gain or loss in the slab, whose commutators,

$$[\hat{F}_L(\omega), \hat{F}_L{}^{\dagger}(\omega')] = [\hat{F}_R(\omega), \hat{F}_R{}^{\dagger}(\omega')] = (1 - |R(\omega)|^2 - |T(\omega)|^2)\delta(\omega - \omega') \tag{3.10}$$

and

$$[\hat{F}_L(\omega), \hat{F}_R{}^{\dagger}(\omega')] = -(R(\omega)T^*(\omega) + T(\omega)R^*(\omega))\delta(\omega - \omega'), \tag{3.11}$$

ensure that both the incoming and outgoing field operators, related by (3.8) and (3.9), have boson commutators similar to (2.3). The noise operators $\hat{F}_L(\omega)$ and $\hat{F}_R(\omega)$ have the characters of destruction operators for attenuated frequencies but they take the natures of creation operators for amplified frequencies. The relations (3.8)–(3.11) apply generally to any amplifying or attenuating beam splitter and they ensure that the system adds the requisite amounts of noise to the output signals (Matloob *et al.* 1997; Barnett *et al.* 1998).

We consider the output on the right of the slab for conditions in which $|L\rangle$ is the vacuum state $|0\rangle_L$. The mean photon-number flux is then determined by the expectation value

$$\langle \hat{b}_R{}^{\dagger}(\omega)\hat{b}_R(\omega')\rangle = G_T(\omega)\langle \hat{a}_R{}^{\dagger}(\omega)\hat{a}_R(\omega')\rangle + \langle \hat{F}_R{}^{\dagger}(\omega)\hat{F}_R(\omega')\rangle. \tag{3.12}$$

The added noise, given by the second term on the right, has the value

$$\langle \hat{F}_R{}^{\dagger}(\omega)\hat{F}_R(\omega')\rangle = \begin{bmatrix} N(\omega, |T|) + 1 \\ -N(\omega, T) \end{bmatrix} \{G_R(\omega) + G_T(\omega) - 1\}\delta(\omega - \omega'), \tag{3.13}$$

where

$$N(\omega, T) = \frac{1}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1}. \tag{3.14}$$

The noise takes its minimum value when the slab effective temperature $|T|$ for amplified frequencies, or temperature T for attenuated frequencies, is zero and this is assumed to be the case in the subsequent calculations. The corresponding noise state is denoted $|0\rangle_F$, and it corresponds to perfect inversion for amplified frequencies and to the zero-temperature ground state for attenuated frequencies. A noise contribution then remains for amplified frequencies but the noise vanishes for attenuated frequencies. Note that, in accordance with the first line of (3.7), the noise output persists even for $G_T(\omega) < 1$ provided that $G_R(\omega) + G_T(\omega) > 1$.

(*b*) *Pulse transmission*

The quantized field theory can be used to study the propagation of non-classical light through the slab. Consider an input where the state $|R\rangle$ is a photon number state (2.4) with the Gaussian form of wavepacket from (2.1) or (2.7). The various measurements that can be made on the right of the slab correspond to expectation values of appropriate functions of the operators $\hat{b}_R(t)$ and $\hat{b}_R{}^{\dagger}(t)$. Some of the results are the same as in the classical theory of pulse propagation, while others show the specific quantum-mechanical properties of the input number state.

The mean output flux, with the flux operator defined as in (2.8), is

$$f(t) = {}_R\langle n(\alpha)|_L\langle 0|_F\langle 0|\hat{b}_R{}^{\dagger}(t)\hat{b}_R(t)|0\rangle_F|0\rangle_L|n(\alpha)\rangle_R. \tag{3.15}$$

The flux operator, obtained by Fourier transformation of (3.12), contains contributions from the transmitted pulse and, for amplified frequencies, from the noise spectrum (3.13). The calculation is straightforward (Artoni & Loudon 1997, 1998) and the contributions of the transmitted pulse are shown in figure 3. The input pulse is assumed to be much longer than the slab thickness, $L \gg 2l$, when the transmission gain has negligible variation over the main frequencies that make up the pulse. The pulse then retains a single component with Gaussian shape on transmission through the slab. Dispersion is neglected and the refractive index, denoted η_{c}, is that at the carrier frequency ω_{c}. Figure 3*a* shows the shift Δx in the position of the peak of the pulse relative to its position at the same instant of time in the absence of the slab, while figure 3*b* shows the ratio of the mean-square length L_{T}^2 of the transmitted pulse to that of the input pulse. The shift in peak position has a component $2l(1 - \eta_{\mathrm{c}})$ from the change in optical path associated with the refractive index of the slab and an oscillatory part caused by the interference of multiply reflected contributions to the output pulse. These produce an additional delay in peak position and a lengthening of the pulse when $4\omega_{\mathrm{c}}\eta_{\mathrm{c}}l/c$ is an even integer multiple of π and the reflected contributions have the same phase as the unreflected output pulse. When $4\omega_{\mathrm{c}}\eta_{\mathrm{c}}l/c$ is an odd integer multiple of π, the successive reflections give rise to contributions of alternating phase, and the corresponding reduction in the strength of the rear of the pulse produces a positive contribution to the peak position and a reduction in the length of the transmitted pulse. Similar effects contribute to the apparent superluminal transmission of pulses through more complicated dielectric systems (Japha & Kurizki 1996). The effects of the multiple reflections are reduced by attenuation and increased by amplification. Of course, as the slab approaches a lasing threshold at frequency ω_{c}, the output tends to a continuous wave, the limit of a pulse of infinite delay and length. Essentially the same distortions as described here occur for an

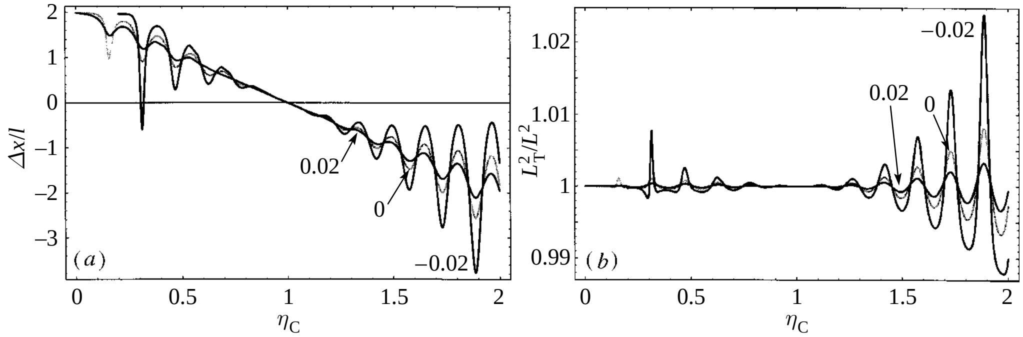

Figure 3. Properties of a Gaussian pulse of input length $L = 40l$ and carrier frequency $\omega_c = 10c/l$ after transmission through a dielectric slab of thickness $2l$ as functions of the refractive index η_c for the extinction coefficients κ_c shown against the curves (*a*) shift in peak position and (*b*) mean-square length.

incident coherent pulse described by (2.15) and more generally in the transmission of pulses described by classical theories.

(*c*) *Antibunching*

In contrast to the essentially classical modifications of the pulse shape, the degree of second-order coherence, given by (2.13) for the input number state, shows specific non-classical features. The degree of second-order coherence of the transmitted beam is obtained from (2.12) with use of the output operator from (3.9) and the noise expectation value from (3.13). The result for zero time delay τ is

$$g^{(2)}_{\text{out}}(t,0) = \frac{\langle \hat{b}_R{}^\dagger(t)\hat{b}_R{}^\dagger(t)\hat{b}_R(t)\hat{b}_R(t)\rangle}{\langle \hat{b}_R{}^\dagger(t)\hat{b}_R(t)\rangle^2} = 2 - \frac{n(n+1)|J_1(t)|^4}{[n|J_1(t)|^2 + J_2]^2}. \tag{3.16}$$

Here the contribution of the transmitted pulse is described by the integral

$$J_1(t) = (2\pi)^{-1/2} \int \mathrm{d}\omega\, T(\omega)\alpha(\omega)\exp(-\mathrm{i}\omega t) \approx T(\omega_c)\tilde{\alpha}(t), \tag{3.17}$$

and that of the noise by

$$J_2 = (2\pi)^{-1} \int \mathrm{d}\omega\, \theta[-\kappa(\omega)]\{G_R(\omega) + G_T(\omega) - 1\}, \tag{3.18}$$

where the unit step function ensures that only amplified frequencies contribute to the noise integral. Transmission through the slab causes no change when only attenuated frequencies are detected ($J_2 = 0$), so that

$$g^{(2)}_{\text{out}}(t,0) = g^{(2)}_{\text{in}}(t,0) = 1 - 1/n, \tag{3.19}$$

and the degree of second-order coherence continues to violate the classical inequality (2.14).

However, the noise contribution does not vanish for amplified frequencies, defined by the first condition in (3.7), and the chaotic value of 2 occurs when the noise dominates the pulse contribution. The latter takes its maximum value for frequencies within a separation c/L from ω_c and times when the peak of the pulse lies within a distance L of the observation point, so that the effects of the noise can be minimized by appropriate filtering of the detected signal. A rough idea of the maximum gain for which the inequality (2.14) is violated can be obtained from the expression in (3.16) by assuming constant gains G_R and G_T across a bandwidth $\Delta\omega$ in the integrals (3.17) and (3.18), with the result

$$nG_T(2G_R + G_T - 2) + (G_R + G_T - 1)^2 < 0. \tag{3.20}$$

In the most favourable case of zero reflection gain, $G_R = 0$, the solution is

$$1 \leqslant G_T < 1 + \left(\frac{n}{n+1}\right)^{1/2}, \tag{3.21}$$

so that only very modest gains of 2 or less are allowed if the non-classical antibunching feature is to be preserved (Loudon & Shepherd 1984).

(*d*) *Squeezing*

Now suppose that the incident state $|R\rangle$ is a squeezed vacuum $|\{\{\beta\}\}\rangle$, while $|L\rangle$ remains in the vacuum state. A homodyne measurement of the field transmitted through the slab is represented by the operator (2.35), with the destruction and creation operators now taken from (3.9) and its Hermitian conjugate. The expectation values needed to evaluate the variance of the homodyne field are readily calculated (Artoni & Loudon 1998), and the result can be written as

$$\begin{aligned}\langle(\Delta E[\phi_{\mathrm{LO}}])^2\rangle_{\mathrm{out}} - 1 = G_T(\omega_{\mathrm{LO}})\{\langle(\Delta E[\phi_{\mathrm{LO}} - \arg T(\omega_{\mathrm{LO}})])^2\rangle_{\mathrm{in}} - 1\} \\ +2\theta[-\kappa(\omega_{\mathrm{LO}})][G_R(\omega_{\mathrm{LO}}) + G_T(\omega_{\mathrm{LO}}) - 1],\end{aligned} \tag{3.22}$$

where the second term on the right is present only for amplified frequencies. The first term on the right contains the homodyne variance of the squeezed input state, given by (2.37), with an additional phase shift from the slab transmission coefficient. This term is negative for appropriate values of the local oscillator phase. The expression (3.22) remains negative for these phase angles and for attenuated frequencies, but the reduction of the homodyne variance of the output field below its coherent value of unity is decreased, and the amount of squeezing approaches zero for heavy attenuation.

The squeezing is more seriously affected for amplified frequencies, where the second term on the right of (3.22) makes a positive contribution that can easily overcome the negative contribution of the first term. Thus for an extremely squeezed input state, with zero homodyne variance, the output homodyne variance is

$$\langle(\Delta E[\phi_{\mathrm{LO}}])^2\rangle = 2G_R(\omega_{\mathrm{LO}}) + G_T(\omega_{\mathrm{LO}}) - 1 \tag{3.23}$$

and the squeezing is entirely lost for gains that satisfy

$$2G_R(\omega_{\mathrm{LO}}) + G_T(\omega_{\mathrm{LO}}) > 2, \tag{3.24}$$

when the output variance is greater than unity. The maximum allowed gain in the most favourable case of $G_R = 0$ is again equal to 2. The squeezing is lost for smaller values of the gains when the amount of input squeezing is less. In contrast, the output variance obtained from (3.22) for an input coherent or vacuum state is

$$\langle(\Delta E[\phi_{\mathrm{LO}}])^2\rangle_{\mathrm{out}} = 2G_R(\omega_{\mathrm{LO}}) + 2G_T(\omega_{\mathrm{LO}}) - 1. \tag{3.25}$$

The unit variance characteristic of the vacuum state is recovered for a frequency that is neither amplified nor attenuated, as in the second condition of (3.7).

4. Conclusions

The descriptions of some of the main examples of pulsed and stationary states of non-classical light have been reviewed. A recently developed quantization scheme for an attenuating or amplifying slab surrounded by empty space has been used to evaluate the effects of transmission through the slab on measures of the non-classicality given by the degree of second-order coherence for direct detection and by the electric-field variance for homodyne detection. Non-classical features tend to survive transmission through an attenuating slab, although their magnitudes may be reduced. The effects of transmission through an amplifying slab are more drastic, and non-classical features are usually lost, even for modest values of the gain. Thus for the phase-insensitive amplification considered here, both the antibunching and squeezing effects are lost for intensity gains with maximum values of order 2.

Any advantages in the lower noise inherent in some kinds of non-classical light are therefore reduced or lost completely after propagation through attenuating or amplifying materials. For example, any initial squeezing is removed after propagation through a few kilometres of even the lowest-loss gain-compensated optical fibres, although the range of reduced-noise signals can be significantly extended by the use of phase-sensitive amplifiers (Mecozzi & Tombesi 1990). Nevertheless, there is little advantage in using such light in long-distance optical communications and useful applications of non-classical light are likely to be restricted to laboratory experiments. Again, in a quite different example covered elsewhere (Samphire *et al.* 1995), the radiation pressure fluctuations at a mirror can be reduced by the use of antibunched light, with possible applications to high-precision interferometry in gravitational wave detection, but the reductions are made negligibly small by the interference of the incident light with vacuum fluctuations.

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